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Revisited Formulation and Applications of FFT Moving Average

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Abstract The fast Fourier transform-moving average (FFT-MA) is an efficient method for the generation of geostatistical simulations. The method relies on the calculation of a filter operator based on the covariance function of interest and the convolution of the filter with a white noise, to generate multiple realizations of spatially correlated variables. In this work, a revisited mathematical formulation of the FFT-MA method, with the exact expression of the filter, is presented. The proposed derivation of the filter is based on the Wiener–Khinchin theorem and the application of the Fourier transform in a discrete domain. In the specific case of white noise, the proposed formulation leads to the same expression of the traditional algorithm. However, the method can be applied to other types of noise. The proposed technique allows the calculation of a specific filter that imposes an exact covariance function on the noise. Therefore, the experimental covariance function is exactly equal to the theoretical one, which is not the case for many common simulation techniques due to the limited sample size. Applications of the FFT-MA method to synthetic and real data sets, including exact interpolation, hard data conditioning and correlated simulations from cross-correlated noises, are also presented.

Keywords FFT moving average · Fast Fourier transform · Geostatistical simulations · Data conditioning

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1 Introduction

Geostatistical simulations are widely used to generate multiple realizations of random fields to represent the distribution of rock and fluid properties and their spatial variability associated to geological stratigraphy and deposition (Armstrong et al. 2003; Gómez-Hernández 2005; Ersoy and Yünsel 2006; Doyen 2007; de Figueiredo et al. 2018). Several techniques can be adopted to simulate stochastic realizations of subsurface models. The most popular algorithm is the sequential Gaussian simulation method (Deutsch and Journel 1992; Journel and Gomez-Hernandez 1993), a geostatistical approach in which the values of the model property are sequentially sampled from a Gaussian distribution with mean and variance equal to the kriging mean and variance computed from the previously simulated values. Several algorithms based on the sequential approach have been presented, including the sequential Gaussian mixture simulation (Grana et al. 2012), the Direct Sequential Simulation (Soares 2001; Tran et al. 2001), and the sequential indicator simulation (Journel and Gomez-Hernandez 1993; Doyen et al. 1994).

In addition to sequential methods, other algorithms have been developed, including the fast Fourier transform-moving average (FFT-MA) (Oliver 1995; Le Ravalec et al. 2000; Doyen 2007), the turning band method (Journel and Huijbregts 1978; Brooker 1985; Hunger et al. 2014), and the Cholesky and lower-upper (LU) decomposition of the covariance matrix methods (Alabert 1987; Davis 1987). In particular, the FFT-MA is a very efficient algorithm for simulating geostatistical realizations in relatively large grids. In general, the FFT-MA is faster than sequential methods. The algorithm is based on the convolution between a filter operator and a white noise; therefore, its implementation is particularly efficient when the convolution is computed in the frequency domain. In order to simulate random fields conditioned by hard data, the FFT-MA can be combined with the probability field simulation (PFS) method (Srivastava 1992). However, this algorithm has two main limitations related to the inaccurate reproduction of local extrema and covariance functions (Pyrcz and Deutsch 2001; Doyen 2007). Despite these limitations, the methodology is widely applied to generate realizations in large grids thanks to its simplicity and efficiency (Le Ravalec-Dupin et al. 2008; Yang and Zhu 2017; de Figueiredo et al. 2018). A more adequate and formally correct approach is the method proposed by Marcotte and Allard (2018), based on post-conditioning by kriging. Indeed, conditional realizations of a Gaussian random field can be obtained in two steps, by generating unconditional realizations and then using kriging to condition the realizations by hard data.

In this work, a revisited derivation of the FFT-MA filter operator is presented. When the noise is white, the same expression of the traditional formulation based on the filter operator is obtained. However, in the proposed formulation, the filter can be applied to other types of noise. In the case of colored noise, the computational cost for the calculation of the filter operator is larger than in case of white noise. The advantage of the proposed formulation is that it allows the calculation of a specific filter that imposes an exact covariance function on the noise. Therefore, the experimental covariance function is exactly equal to the theoretical one. In traditional simulation techniques, experimental and theoretical covariance functions are not necessarily the same due to the limited size of the simulation grid.

Applications of the method to simulations conditioned by hard data and simulations of correlated variables are presented using the traditional FFT-MA. In particular, realizations conditioned by hard data are obtained by solving a linear system of equations for the components of the white noise at the locations of the measured data. A mathematical exact interpolator is also obtained by calculating the mean of the conditional realizations. The generation of correlated simulations is obtained by applying the filter operator to correlated white noises.

Section 2 reviews the previous formulations of the FFT-MA method presented in literature, describes the mathematical details of the revised formulation for one variable, and presents the formulation of the methodology for hard data conditioning and correlated simulations. Section 3 shows the results of the application to synthetic and real data sets.

2 Methodology

2.1 Literature Review

The FFT-MA method for geostatistical simulations was first introduced by Oliver (1995). The filter operator is obtained by defining the spatially correlated random field $y(\mathbf{x})$ as the convolution of the operator filter $f(\mathbf{x})$ and the white noise $z(\mathbf{x})$

$$y(\mathbf{x}) = f(\mathbf{x}) * z(\mathbf{x}) = \int_U f(\mathbf{x}-\mathbf{x}')z(\mathbf{x}')\mathrm{d}\mathbf{x}'. \tag{1}$$

By definition, the covariance function of $y(\mathbf{x})$ is

$$\begin{aligned} c(\Delta\mathbf{x}) &= E\{y(\mathbf{x})y(\mathbf{x}+\Delta\mathbf{x})\} \\ &= E\left\{\int_U f(\mathbf{x}-\mathbf{x}')z(\mathbf{x}')\mathrm{d}\mathbf{x}' \int_U f(\mathbf{x}+\Delta\mathbf{x}-\mathbf{x}'')z(\mathbf{x}'')\mathrm{d}\mathbf{x}''\right\}. \end{aligned} \tag{2}$$

Assuming that the mean operator can be applied to several samples of $y(\mathbf{x})$, the covariance function becomes

$$c(\Delta\mathbf{x}) = \int_U \int_U f(\mathbf{x}-\mathbf{x}')f(\mathbf{x}+\Delta\mathbf{x}-\mathbf{x}'')E\left\{z(\mathbf{x}')z(\mathbf{x}'')\right\}\mathrm{d}\mathbf{x}'\mathrm{d}\mathbf{x}''. \tag{3}$$

If the noise $z(\mathbf{x})$ is white, then

$$E\left\{z(\mathbf{x}')z(\mathbf{x}'')\right\} = \sigma_z^2\ \delta(\mathbf{x}'-\mathbf{x}''), \tag{4}$$

where σ_z^2 is the noise variance. Then, the covariance function can be written as

$$c(\Delta\mathbf{x}) = \sigma_z^2 \int_U f(\mathbf{x}-\mathbf{x}')f(\mathbf{x}+\Delta\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x}'. \tag{5}$$

Because the convolution is commutative with respect to translations and the filter operator is real and even, the relation between the filter and the covariance function is obtained as

$$c(\Delta \mathbf{x}) = \sigma_z^2 \; f(\mathbf{x}) * f(\mathbf{x}). \tag{6}$$

In the frequency domain, the convolution is the product of the functions; therefore, Eq. (6) is

$$C(\mathbf{w}) = \sigma_z^2 |F(\mathbf{w})|^2. \tag{7}$$

A different formulation proposed by Le Ravalec et al. (2000) is based on the Wiener–Khinchin theorem (Wiener 1966). In this formulation, the expectation of the autocovariance function $C(w)$ of the sample $Y(w)$, in the frequency domain, is

$$C(w) = E\left\{Y(w)Y^*(w)\right\}. \tag{8}$$

Bacause $y(t)$ is defined as a convolution in the frequency domain, the Fourier transform $Y(w)$ of $y(t)$ can be expressed as $Y(w) = F(w)Z(w)$. Hence

$$C(w) = E\left\{F(w)Z(w)F^*(w)Z^*(w)\right\}. \tag{9}$$

By assuming that the expectation is calculated over several realizations, $C(w)$ can be rewritten as

$$C(w) = F(w)F^*(w)E\left\{Z(w)Z^*(w)\right\}. \tag{10}$$

Because $z(t)$ is a stationary random field associated with a Dirac covariance function (i.e., white noise), then $C(w)$ can be obtained as

$$C(w) = F(w)F^*(w)\sigma_z^2, \tag{11}$$

where σ_z^2 is the total variance of $z(t)$. Finally, because the filter is even, $C(w)$ can be then written as

$$C(w) = \sigma_z^2 |F(w)|^2. \tag{12}$$

2.2 Revisited Formulation

By applying the Wiener–Khinchin theorem in the discrete form (Wiener 1966), the autocovariance function C_w of y_t can be written in the frequency domain as

$$C_w = \frac{1}{N} Y_w \, Y_w^*, \tag{13}$$

where Y_w is the convolution of the filter f_t and the noise z_t

$$Y_w = F_w Z_w, \tag{14}$$

where F_w and Z_w are the discrete Fourier transforms of f_t and z_t, respectively. By using the definition of discrete Fourier transform, Y_w can be rewritten as

$$Y_w = F_w \sum_t z_t e^{-iwt}. \tag{15}$$

Then, by replacing the discrete Fourier transform of Y_w in Eq. (13) and by using the even property of the filter ($F_w F_w^* = |F_w|^2$), C_w is obtained as

$$C_w = \frac{|F_w|^2}{N} \sum_t \sum_{t'} z_t z_{t'} e^{iw(t'-t)}. \tag{16}$$

The expression of C_w can be split into two terms

$$C_w = \frac{|F_w|^2}{N} \left[\sum_t z_t^2 + 2 \sum_{t=0}^{N} \sum_{t'=t+1}^{N} z_t z_{t'} e^{iw(t'-t)} \right], \tag{17}$$

where the first term is associated to the product of the components with the same indexes, and the second term is associated to different components.

By introducing the variable Δt, defined as $\Delta t = t' - t$, and replacing this expression in Eq. (17), C_w can be written as

$$C_w = \frac{|F_w|^2}{N} \left[\sum_t z_t^2 + 2 \sum_{t=0}^{N} \sum_{\Delta t=1}^{N-t} z_t z_{\Delta t+t} e^{iw\Delta t} \right], \tag{18}$$

or

$$C_w = |F_w|^2 \left[\frac{1}{N} \sum_t z_t^2 + 2 \sum_{\Delta t=1}^{N-t} e^{iw\Delta t} \left(\frac{1}{N} \sum_{t=0}^{N} z_t z_{\Delta t+t} \right) \right]. \tag{19}$$

In Eq. (19), the term highlighted in parenthesis is the definition of the autocovariance function of the random noise z_t. In the particular case where z_t is white noise, the term in parenthesis is zero for any $\Delta t > 0$. Furthermore, the first summation in Eq. (19) is the noise variance; therefore, Eq. (19) becomes

$$C_w = \sigma_z^2 |F_w|^2. \tag{20}$$

From Eq. (19), the expression of the filter weights F_w can be derived as

$$F_w = \sqrt{C_w \left[\frac{1}{N} \sum_t z_t^2 + 2 \sum_{\Delta t=1}^{N-t} e^{iw\Delta t} \left(\frac{1}{N} \sum_{t=0}^{N} z_t z_{\Delta t+t} \right) \right]^{-1}}. \tag{21}$$

The complexity of the computation is of order $O(N^3)$ compared to the traditional FFT-MA of order $O(N \ln(N))$, for the specific case of white noise (Eq. 20). However, Eq. (21) can be applied to any type of noise. Furthermore, even in the white noise case, it is unlikely to obtain perfect white noise using standard random sampling techniques for generating realizations of random fields in a grid with limited size (e.g., smaller than 10^4 elements). If Eq. (21) is calculated for a specific noise, the convolution with the exact filter leads to a correlated random field with an experimental covariance function (Eq. 13) exactly equal to the theoretical one. The exact reproduction of the covariance function is guaranteed in the frequency domain but not in the original spatial domain. This is not the case for traditional methods when applied to simulations of random fields in small grids. In some applications, reproducing the exact theoretical covariance function might be a limitation, because it might be desirable to obtain realizations whose experimental covariance accounts for variations around the theoretical model, for example, for large lags of the spatial covariance function. Both the traditional FFT-MA and the exact formulation might introduce a periodic behavior in the realizations due to the use of the Fourier transform, as documented in Le Ravalec et al. (2000).

The traditional definition of the filter (Eq. 20) leads to a different interpretation of the usual moving average operators. In fact, by applying the discrete inverse Fourier transform to C_w at $t = 0$, the realization variance σ_y^2 can be obtained as

$$c_{t=0} = \frac{\sigma_z^2}{N} \sum_w |F_w|^2 = \sigma_y^2. \tag{22}$$

Based on Parseval's theorem in the discrete domain (Arfken et al. 2011), the sum on the right-hand side of Eq. (22) can be expressed as

$$\sum_t |f_t|^2 = \frac{1}{N} \sum_w |F_w|^2. \tag{23}$$

By applying this expression to Eq. (22), the following equivalence is obtained

$$\sum_t |f_t|^2 = \frac{\sigma_y^2}{\sigma_z^2}. \tag{24}$$

Therefore, the so-obtained filter operator differs from the traditional moving average operator, in which the summation over all its components is equal to one. It is also possible to find a filter where the summation over its components is zero.

2.3 Correlated Simulations

Assuming two white noises z^1 and z^2 with variance σ_z^2 and correlation ρ^{z^1,z^2}, it is possible to generate two spatially correlated realizations y^1 and y^2 by applying a filter operator. In this case, the cross-covariance function of the realizations in the frequency domain is

$$C_w^{y_1,y_2} = \frac{Y_w^1\, Y_w^{2*}}{N} = \frac{F_w\, F_w^*\, Z_w^1\, Z_w^{2*}}{N}, \tag{25}$$

or

$$C_w^{y_1,y_2} = |F_w|^2\, C_w^{z_1,z_2}, \tag{26}$$

where $C_w^{z_1,z_2} = \sigma_z^2 \rho^{z_1,z_2}$.

The cross-covariance at lag zero can be calculated by applying the discrete inverse Fourier transform for $t = 0$

$$c_{t=0}^{y_1,y_2} = \frac{1}{N} \sum_w |F_w|^2\, \sigma_z^2 \rho^{z_1,z_2}. \tag{27}$$

where, by definition, $c_{t=0}^{y_1,y_2}$ is equal to the total covariance $\sigma_y^2 \rho^{y_1,y_2}$ between the realizations y^1 and y^2. By applying Parseval's theorem in the discrete domain to Eq. (27), the following expression is obtained

$$\sigma_y^2 \rho^{y_1,y_2} = \sigma_z^2 \rho^{z_1,z_2} \sum_t |f_t|^2. \tag{28}$$

By combining Eq. (28) with Eq. (24), it is straightforward to obtain

$$\rho^{y_1,y_2} = \rho^{z_1,z_2}. \tag{29}$$

This result shows that the correlation between white noises is preserved after the filtering process. As a consequence, it is possible to generate multiple correlated simulations by applying the same filter to multiple correlated white noises. In practical applications, the white noises can be generated using a point-wise bivariate sampling approach. Other techniques for the simulation of correlated realizations using the FFT-MA method can be found in Le Ravalec-Dupin and Da Veiga (2011) and Liang et al. (2016). For multivariate problems, another application is the simulation of corregionalizations with variables with different structures, using different filters applied to correlated noises (Bourgault and Marcotte 1991, 1993).

It is also possible to apply score transformations to Gaussian distributions to obtain non-Gaussian simulations; however, because the transformation is nonlinear, the final covariance function of the realization may not be preserved (Wackernagel 2003).

2.4 Hard Data Conditioning

In several applications, the probability field simulation (PFS) method is used to condition the FFT-MA simulations to available measured data. However, this algorithm can lead to undesired artifacts, including inaccurate predictions of local extrema and of the covariance function (Pyrcz and Deutsch 2001; Doyen 2007). Alternatively, unconditional realizations of a Gaussian random field could be generated and post-conditioned by kriging as in Marcotte and Allard (2018).

In this work, a new approach is presented, based on a technique originally proposed by Alabert (1987) and Davis (1987) to produce conditional simulations via LU decomposition of the covariance matrix. The approach is applied in a time domain, in which the noise components are locally perturbed to ensure that the realizations honor the data after applying the convolution operator.

The convolution of the filter and the noise, in a time domain, can be written in the matrix form

$$\mathbf{y} = \mathbf{T}\mathbf{z}, \tag{30}$$

where $\mathbf{T}$ is the convolution matrix generated by the discrete filter $\mathbf{f}$, and $\mathbf{z}$ is the discrete white noise. In practical applications, the discrete filter $\mathbf{f}$ is obtained by applying the inverse fast Fourier transform (IFFT) of the filter in the frequency domain (Eqs. 12 and 20).

To honor the measured data, the specific field components of the realizations $\mathbf{y}$ must be equal to the data $\mathbf{d}$, at the data locations. The rows of the product in Eq. (30) can be rearranged such that the first row corresponds to the hard data locations

$$\begin{pmatrix} \mathbf{y}_d \\ \mathbf{y}_n \end{pmatrix} = \begin{pmatrix} \mathbf{T}_d & \mathbf{T}_o \\ \mathbf{T}_1 & \mathbf{T}_2 \end{pmatrix} \begin{pmatrix} \mathbf{z}_d \\ \mathbf{z}_o \end{pmatrix} = \begin{pmatrix} \mathbf{d} \\ \mathbf{y}_n \end{pmatrix}. \tag{31}$$

The solution of the system at the data locations can then be obtained as

$$\mathbf{z}_d = \mathbf{T}_d^{-1} \left(\mathbf{d} - \mathbf{T}_o \mathbf{z}_o \right). \tag{32}$$

By applying the convolution operator (Eq. 30), the realization $\mathbf{z}$ is conditioned by the measured data $\mathbf{d}$.

The computational cost of conditional simulations is larger than the cost of unconditional simulations. Indeed, the formulation of the algorithm for hard data conditioning is expressed in the time domain, rather than the frequency domain, and it requires the calculation of the convolution matrix. Such matrix can be very large for three-dimensional problems. The computational cost and the memory requirements are larger than the approach proposed in Marcotte and Allard (2018), based on FFT-MA unconditional simulations and post-conditioning by kriging.

To prove that the proposed formulation leads to an exact estimator, it is possible to compute the mean of several realizations $\mathbf{y}$

$$E\{\mathbf{y}\} = \mathbf{T} E\{\mathbf{z}\}, \tag{33}$$

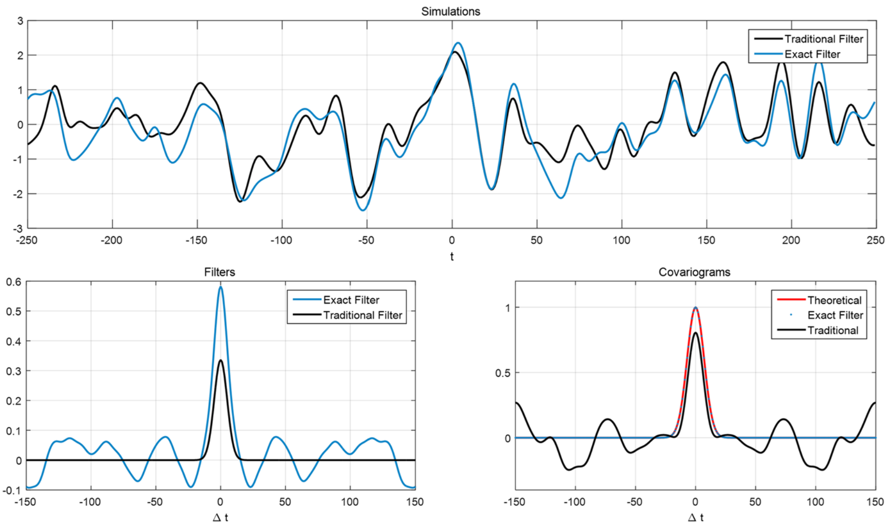

Fig. 1 Example of application to a Gaussian covariance function model. Top: simulation results with traditional filter (black) and exact filter (blue). Bottom left: comparison of the filters, traditional formulation (black) and exact filter of the proposed formulation (blue). Bottom right: experimental covariance function obtained with traditional filter (black) and exact filter (blue)

which can be rearranged as in Eq. (31)

$$E\{\mathbf{y}\} = \mathbf{T}\begin{pmatrix} E\{\mathbf{z}_d\} \\ E\{\mathbf{z}_o\} \end{pmatrix}. \tag{34}$$

The mean of the noise components $\mathbf{z}_d$ at the data locations can be calculated using Eq. (32)

$$E\{\mathbf{z}_d\} = \mathbf{T}_d^{-1}\left(\mathbf{d} - \mathbf{T}_o E\{\mathbf{z}_o\}\right) = \mathbf{T}_d^{-1}\mathbf{d}. \tag{35}$$

Because the noise is assumed to be white, the mean of the noise components at other locations is $\mathbf{0}$. Then, the exact estimator is given by

$$E\{\mathbf{y}\} = \mathbf{T}\begin{pmatrix} \mathbf{T}_d^{-1}\mathbf{d} \\ \mathbf{0} \end{pmatrix}. \tag{36}$$

The exact conditioning property is valid for each realization.

One of the limitations of the proposed method is related to the memory requirements that make FFT-MA not applicable to very large grids with large correlation ranges. For large three-dimensional applications, other simulation methods such as the turning band method (Journel and Huijbregts 1978; Brooker 1985; Hunger et al. 2014) are more suitable.

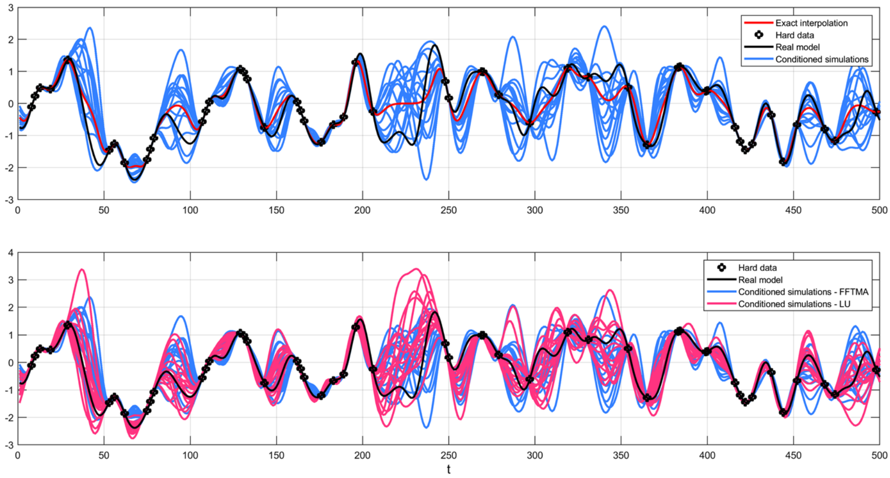

Fig. 2 Example of application of conditional FFT-MA. Top: Conditional sampling given a set of hard data; the reference model is shown in black, the conditional realizations in blue, and the exact interpolation in red. Bottom: comparison of several realizations obtained from the proposed conditional FFT-MA (blue) and from conditional LU decomposition (magenta)

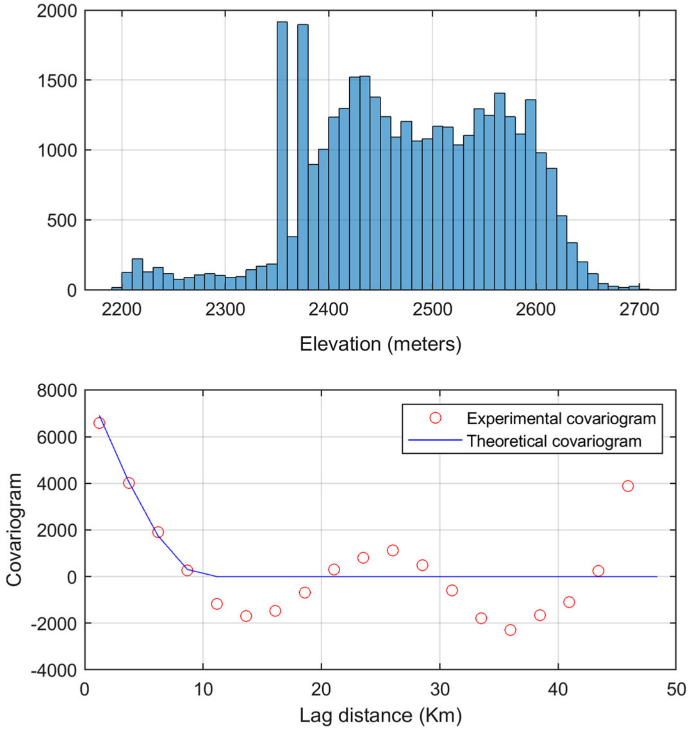

Fig. 3 Elevation data in meters of a part of Yellowstone Park. Top: Elevation histogram; bottom: experimental covariance function of the elevation (red points) fitted by a spherical theoretical model with a range of 12 km and sill of 8943 m^2 (blue curve)

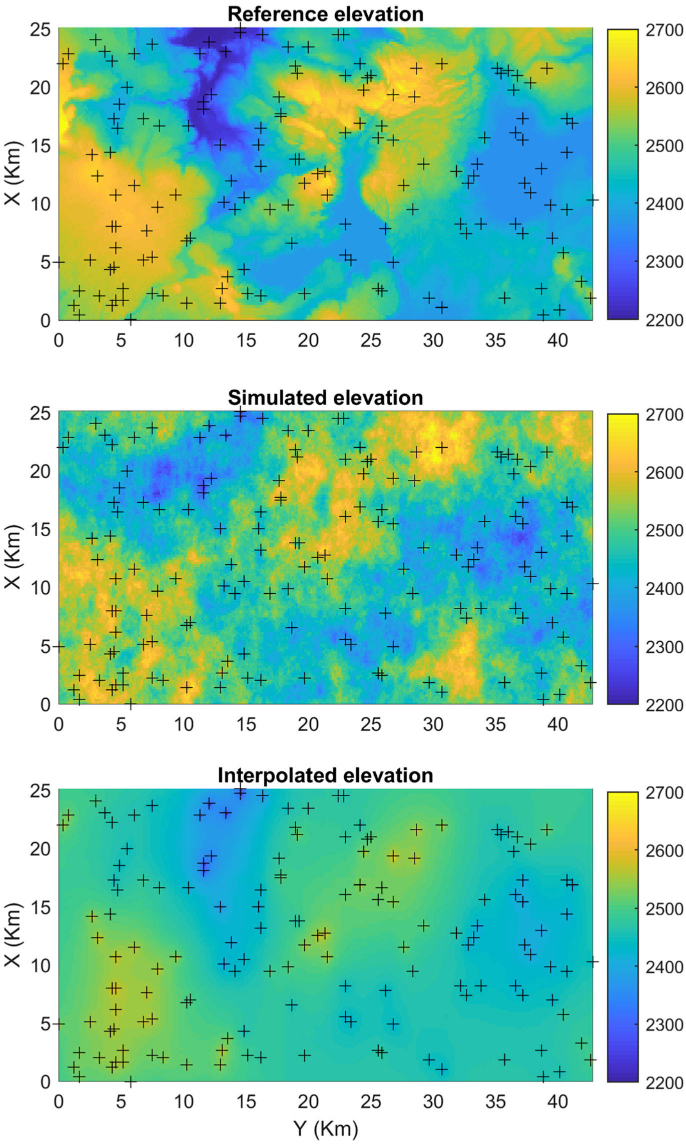

Fig. 4 Elevation data in meters of a part of Yellowstone Park. Top: reference model (black crosses represent hard data locations. Middle: conditional simulated model. Bottom: interpolated model

3 Applications

3.1 Univariate Examples

To validate the revisited formulation, a one-dimensional Gaussian covariance function model is adopted. In the first example, a convolutional operator is applied to a white

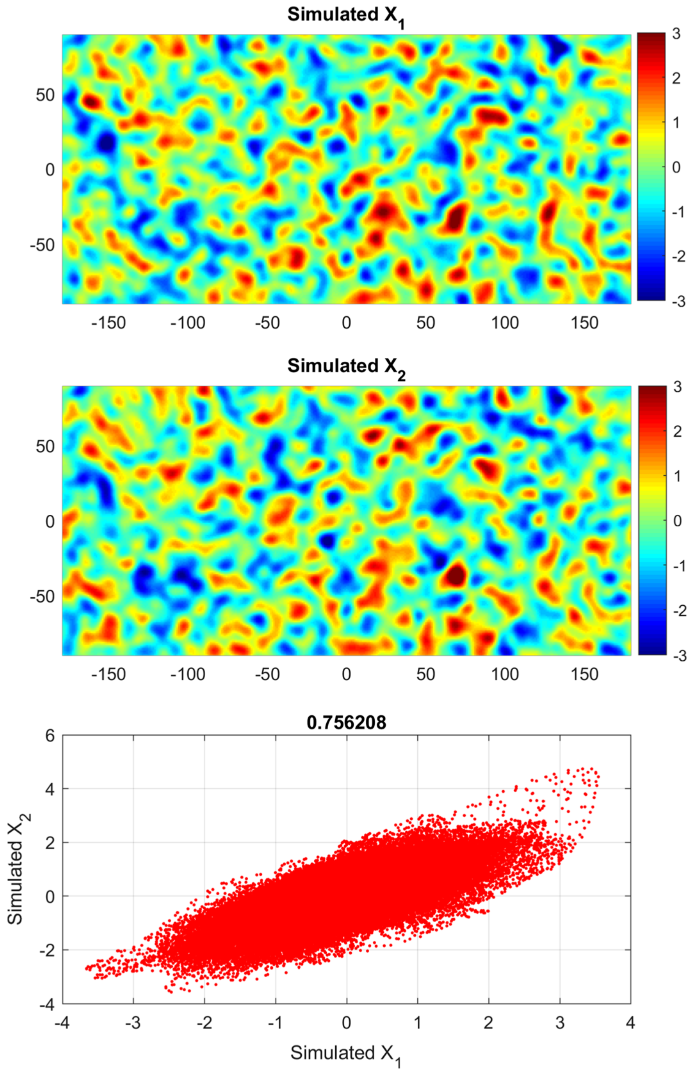

Fig. 5 Correlated simulation. Top and middle plots: two realizations with a correlation of 0.75 and Gaussian covariance model. Bottom plot: cross-plot of the simulated values with an experimental correlation of 0.756208

noise using the traditional FFT-MA filter (Eq. 7) and the proposed exact filter (Eq. 19), respectively. Figure 1 shows the filters and the corresponding realizations generated by each filter. The realizations show similar results; however, the experimental covariance function (Fig. 1) obtained with the exact filter is exactly equal to the theoretical model, whereas the experimental covariance obtained with the traditional formulation differs from the theoretical one due to the limited number of samples. The experimental covariance functions are calculated using Eq. (13) for the entire simulated field.

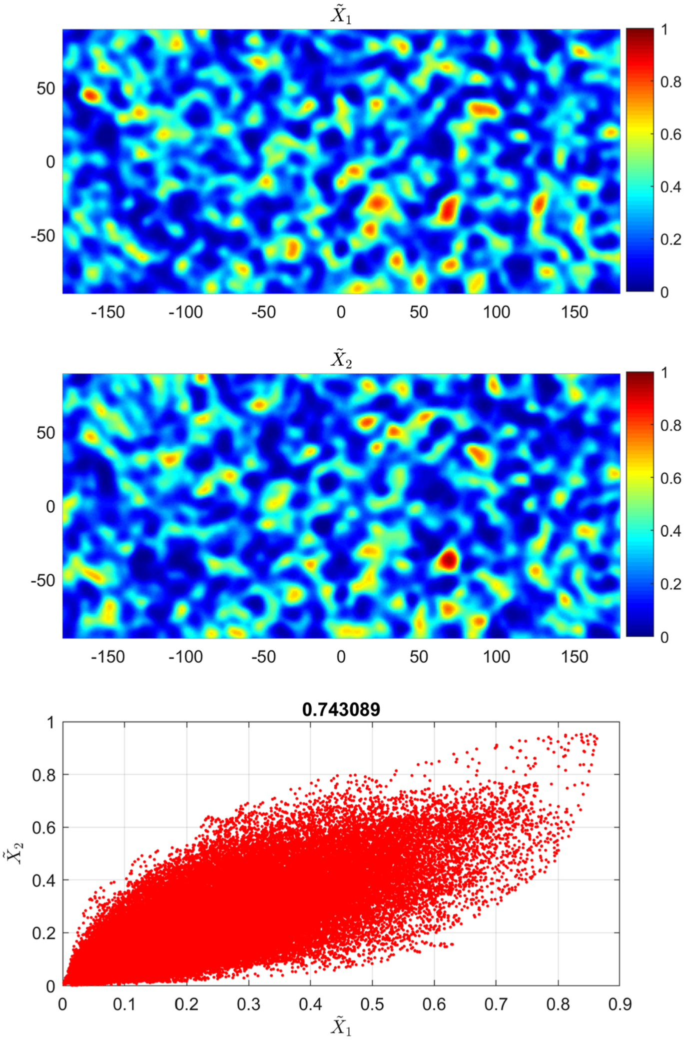

Fig. 6 Non-Gaussian simulation. Top and middle plots: realizations of variables X_1 and X_2 following the target beta distribution with the Gaussian covariance model. Bottom plot: cross-plot of the simulated values with experimental correlation 0.743089

Hard data conditioning and exact interpolation are demonstrated through a one-dimensional application, in which a random realization is generated using a theoretical covariance function and is used as a reference model to extract hard data. Figure 2 (top) shows the one-dimensional reference model with several realizations conditioned by the hard data. The realizations are obtained using Eqs. (30) and (32). The exact interpolation is computed using Eq. (36). Figure 2 (bottom) compares multiple realizations obtained with the proposed FFT-MA for conditional simulations with realizations obtained with the LU decomposition approach, showing similar results.

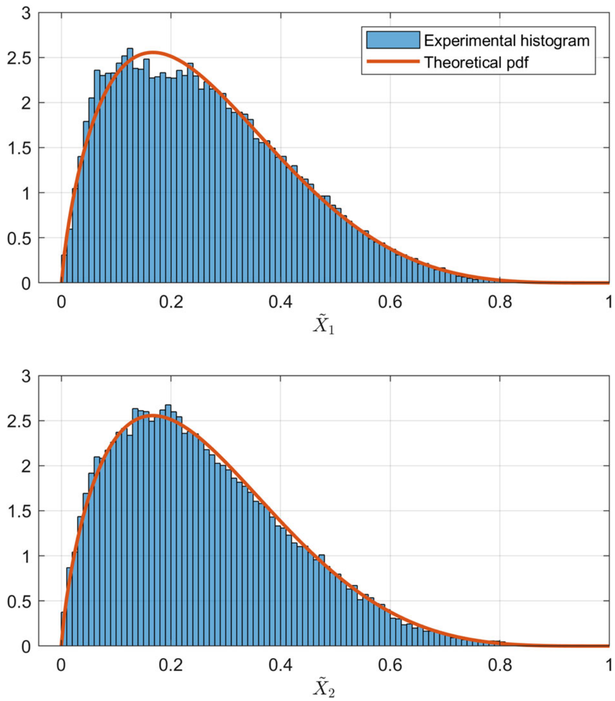

Fig. 7 Non-Gaussian simulation: histogram of the non-Gaussian realizations. The target beta distribution is shown in orange

3.2 Multivariate Examples

The proposed methodology has been tested in a real two-dimensional application representing the elevation (in meters) of a region of Yellowstone Park. The selected area of the park is located approximately between Old Faithful and the west side of Yellowstone Lake, and corresponds to a latitude of approximately 44° and a longitude of approximately $-110°$. The histogram of the data is shown in Fig. 3. The experimental covariance function is calculated assuming transverse isotropy, and it can be fitted by a theoretical spherical model with a range of 12 km and sill of 8943 m^2.

The conditioning hard data is generated by randomly sampling 150 data points of the reference elevation data, shown by the black points in Fig. 4 (top). The results of the simulation are shown in Fig. 4 (middle) and (bottom), where a conditional realization and the exact interpolation (Eq. 36) are compared to the reference model. Both simulation and interpolation are conditioned by the hard data.

To illustrate the method for the simulation of multiple correlated properties, a two-dimensional synthetic example is introduced (Fig. 5). Two spatially correlated Gaussian realizations (X_1 and X_2) are obtained by applying the filter operator to two white noises with a correlation of 0.75. The cross-plot of the simulated values of the correlated realizations X_1 and X_2 shows that the correlation is approximately the same (0.756208).

Similarly, the application of the methodology to non-Gaussian distributions is illustrated using the same two-dimensional synthetic example. The target distribution corresponds to a beta probability density function with parameters $A = 1.8$ and $B = 5$, for the random realizations X_1 and X_2 introduced in the previous example. The non-Gaussian realizations are shown in Fig. 6. The cross-plot of the simulated values shows that in this example, the correlation between the variables is approximately preserved and equal to 0.743089, after the transformation. In general, the correlation depends on the type of transformation. For strongly nonlinear transformations of the Gaussian variables, the correlation might not be exactly preserved. The histograms of the simulated values are displayed in Fig. 7 and show that the transformations are properly applied to the variables. Indeed, the experimental distributions match the target beta probability density function.

4 Conclusions

In this work, a new formulation for the derivation of the operator filter of the FFT-MA method has been presented and demonstrated through several applications. In the case of white noise, the proposed formulation is equivalent to the traditional approach available in literature. The revisited formulation can also be applied to other types of noise and allows the computation of a specific filter associated to the noise. The proposed formulation generates multiple realizations with an experimental covariance function exactly equal to the theoretical covariance model. An approach for generating conditional realizations given a set of hard data measurements has also been proposed. In this approach, an exact interpolation method is obtained by calculating the mean over the conditional simulations. Additional applications of the method include examples of correlated realizations of multiple variables and simulations using non-Gaussian distributions. The proposed formulation and methodology have been validated through one-dimensional illustrative examples and two-dimensional applications to synthetic and real data.

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